The Christoffel problem and two analogs of the Minkowski problem in Riemannian space.

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Abstract

Author finds the solutions of the Christoffel problem for open and closed surfaces in Riemannian space. The Christoffel problem is reduced to the problem of construction the continuous G-deformations preserving the sum of principal radii of curvature for every point of surface in Riemannian space. G-deformation transfers every normal vector of surface in parallel along the path of the translation for each point of surface. The following analogs of the Minkowski problem for open and closed surfaces in Riemannian space are being considered in this article: 1) the problem of construction the surface with prescribed mean curvature and condition of G-deformation; 2) the problem of construction the deformations preserving the area of each arbitrary region of surface and condition of G-deformation.

Introduction

The Christoffel problem (ChP) is well known fundamental problem of differential geometry. Author solves the ChP in Riemannian space as the problem of finding the continuous $G-$deformations with prescribed the sum of principal radii of curvature.

In the article, there is being considered the problem of construction the surface with prescribed mean curvature and condition of G-deformation in Riemannian space, which is the analog of the Minkowski problem.

The second analog of the Minkowski problem is finding the deformations preserving the area of each arbitrary region of surface with condition of $G-$deformation.

Theorems 1 and 2 represents the properties of solutions of considered problems for open and closed surfaces in Riemannian space respectively.

§1.1. Basic definitions. Statement of the main results for open surfaces in Riemannian space.

Let R^3 be the three-dimensional Riemannian space with metric tensor $\tilde{a}_{\alpha\beta}$, F^+ be the two-dimensional simply connected oriented surface in R^3 with the boundary ∂F.

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Let $F^+ \in C^{m,\nu}, \nu \in (0;1), m \geq 4$. $\partial F \in C^{m+1,\nu}$. Let F^+ has all strictly positive principal curvatures k_1 and k_2. Let F^+ be oriented so that mean curvature H is strictly positive. Denote $K = k_1 k_2$.

Let F^+ be given by immersion of the domain $D \subset E^2$ into R^3 by the equation: $y^\sigma = f^{\sigma+}(x), x \in D, f : D \to R^3$. Denote by $d\sigma(x) = \sqrt{g} dx^1 \wedge dx^2$ the area element of the surface F^+. We identify the points of immersion of surface F^+ with the corresponding coordinate sets in R^3. Without loss of generality we assume that D is unit disk. Let x^1, x^2 be the Cartesian coordinates.

Symbol $_{,i}$ denotes covariant derivative in metric of surface F^+. Symbol ∂_i denotes partial derivative by variable x^i. We will assume $\dot{f} \equiv \frac{df}{dt}$. We define $\Delta(f) \equiv f(t) - f(0)$.

We consider continuous deformation of the surface F^+: $\{F_t\}$ defined by the equations

$$y^\sigma_t = y^\sigma + z^\sigma(t), z^\sigma(0) \equiv 0, t \in [0; t_0], t_0 > 0. \tag{1.1}$$

Definition 1 . *Deformation $\{F_t\}$ is called the continuous deformation preserving the sum of principal radii of curvature (or $Ch-$deformation) if the following condition holds: $\Delta(\frac{1}{k_1} + \frac{1}{k_2}) = 0$ and $z^\sigma(t)$ is continuous by t, where k_1 and k_2 are principal curvatures of F^+.*

Definition 2 . *Deformation $\{F_t\}$ is called the continuous deformation preserving the mean curvature (or $H-$deformation) if the following condition holds: $\Delta(H) = 0$ and $z^\sigma(t)$ is continuous by t.*

Definition 3 . *Deformation $\{F_t\}$ is called the continuous $A-$deformation if the following condition holds: $d\sigma_t - d\sigma = 0$ and $z^\sigma(t)$ is continuous by t.*

This means that $A-$deformation preserves the area of each arbitrary region of surface.

The deformation $\{F_t\}$ generates the following set of paths in R^3

$$u^{\alpha_0}(\tau) = (y^{\alpha_0} + z^{\alpha_0}(\tau)), \tag{1.2}$$

where $z^{\alpha_0}(0) \equiv 0, \tau \in [0; t], t \in [0; t_0], t_0 > 0$.

Definition 4 . *The deformation $\{F_t\}$ is called the $G-$deformation if every normal vector of surface transfers in parallel along the path of the translation for each point of surface.*

Indices denoted by Greek alphabet letters define tensor coordinates in Riemannian space R^3. We use the following rule: a formula is valid for all admissible values of indices if there are no instructions for which values of indices it is valid. We use the Einstein rule. Let g_{ij} and b_{ij} be the coefficients of the first and the second fundamental form respectively.

Let, along the ∂F, be given vector field tangent to F^+. We denote it by the following formula:

$$v^\alpha = l^i y^\alpha_{,i}. \tag{1.3}$$

We consider the boundary-value condition:

$$\tilde{a}_{\alpha\beta} z^\alpha v^\beta = \tilde{\gamma}(s,t), s \in \partial D. \tag{1.4}$$

Let v^α and $\tilde{\gamma}$ be of class $C^{m-2,\nu}$.

We denote:

$$\tilde{\lambda}_k = \tilde{a}_{\alpha\beta} y^\alpha_{,k} v^\beta, k = 1, 2. \tag{1.5}$$

$$\lambda_k = \frac{\tilde{\lambda}_k}{(\tilde{\lambda}_1)^2 + (\tilde{\lambda}_2)^2}, k = 1, 2. \tag{1.6}$$

$$\lambda(s) = \lambda_1(s) + i\lambda_2(s), s \in \partial D. \tag{1.7}$$

Let n be the index of the given boundary-value condition

$$n = \frac{1}{2\pi} \Delta_{\partial D} \arg \lambda(s). \tag{1.8}$$

Theorem 1 . *Let $F^+ \in C^{m,\nu}, \nu \in (0;1), m \geq 4$, $\partial F \in C^{m+1,\nu}$. Let $\tilde{a}_{\alpha\beta} \in C^{m,\nu}$, $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$. Let $v^\beta, \tilde{\gamma} \in C^{m-2,\nu}(\partial D)$, $\tilde{\gamma}$ is continuously differentiable by t. Let, at the point $(x^1_{(0)}, x^2_{(0)})$ of the domain D, the following condition holds: $\forall t : z^\sigma(t) \equiv 0$.*

Then the following statements hold:

1) if $n > 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon$ for all $t \in [0, t_0)$:

1a) there exists $(2n-1)-$parametric $ChG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

1b) there exists $(2n-1)-$parametric $HG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

1c) there exists $(2n-1)-$parametric $AG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

2) if $n < 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon(t_0)$ for all $t \in [0, t_0)$:

2a) there exists at most one $ChG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

2b) there exists at most one $HG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

2c) there exists at most one $AG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

3) if $n = 0$ then there exist $t_0 > 0$ and $\varepsilon(t_0) > 0$ such that for any admissible $\tilde{\gamma}$ satisfying the condition: $|\dot{\tilde{\gamma}}|_{m-2,\nu} \leq \varepsilon$ for all $t \in [0, t_0)$:

3a) there exists one $ChG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

3b) there exists one $HG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

3c) there exists one $AG-$deformation of class $C^{m-2,\nu}(\bar{D})$ continuous by t.

§1.2. Statement of the main results for closed surfaces in Riemannian space.

Let F be the two-dimensional simply connected oriented closed surface in R^3.

Let $F \in C^{m,\nu}, \nu \in (0;1), m \geq 4$. Let F has all strictly positive principal curvatures k_1 and k_2. Let F be oriented so that mean curvature H is strictly positive.

Let F be glued from the two-dimensional simply connected oriented surfaces F^+ and F^- of class $C^{m,\nu}$. Let F^+ be attached to F^- along the common boundary ∂F of class $C^{m+1,\nu}$.

Let F^+ and F^- be given by immersions of the domain $D \subset E^2$ into R^3 by the equation: $y^\sigma = f^{\sigma\pm}(x), x \in D,\ f^\pm : D \to R^3$.

Theorem 2 . *Let $F \in C^{m,\nu}, \nu \in (0;1), m \geq 4$, be closed surface. Let F be glued from the two-dimensional simply connected oriented surfaces F^+ and F^- of class $C^{m,\nu}$. Let F^+ be attached to F^- along the common boundary ∂F of class $C^{m+1,\nu}$. Let $\tilde{a}_{\alpha\beta} \in C^{m,\nu}$, $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.*

1) Then there exists $t_0 > 0$ such that for all $t \in [0, t_0)$:

1a) there exists three-parametric $ChG-$deformation of class $C^{m-2,\nu}$ continuous by t.

1b) there exists three-parametric $HG-$deformation of class $C^{m-2,\nu}$ continuous by t.

1c) there exists three-parametric $AG-$deformation of class $C^{m-2,\nu}$ continuous by t.

2) If, at the point $T_0 \in F^+$, the following additional condition holds: $\forall t : z^\sigma(t) \equiv 0$. Then there exists $t_0 > 0$ such that for all $t \in [0, t_0)$:

2a) there exists only zero $ChG-$deformation of class $C^{m-2,\nu}$ continuous by t.

2b) there exists only zero $HG-$deformation of class $C^{m-2,\nu}$ continuous by t.

2c) there exists only zero $AG-$deformation of class $C^{m-2,\nu}$ continuous by t.

We use all designations from [30, 32].

§2. Deduction the formulas of $ChG-$deformations, $HG-$deformations and $AG-$deformations for surfaces in Riemannian space.

§2.1. The formulas of $G-$deformations, $\Delta(g)$ and $\Delta(k_1k_2)$.

We denote:

$$z^\sigma(t) = a^j(t)y^\sigma{}_{,j} + c(t)n^\sigma, \tag{2.1.1}$$

where $a^j(0) \equiv 0, c(0) \equiv 0$, n^σ is unit normal vector of surface at the point (y^σ). Therefore the deformation of surface is defined by functions a^j and c. We introduce conjugate isothermal coordinate system where $b_{ii} = V, i = 1, 2, b_{12} = b_{21} = 0$.

The equations of $G-$deformation were obtained in [30] and [32]:

$$\partial_2\dot{a}^1 - \partial_1\dot{a}^2 + p_k\dot{a}^k = \dot{\Psi}_1, \tag{2.1.2}$$

where p_k and $\dot{\Psi}_1$ are defined in [30]. Note that p_k do not depend on t.

The function $\dot{c}$ is found on functions $\dot{a}^i$ from formulas obtained in [30] and [32].

We have from [30]:

$$\Delta(g) = 2g(\partial_1 a^1 + \partial_2 a^2 + q_k a^k - \Psi_2), \tag{2.1.3}$$

where

$$q_1 = \partial_1(\ln\sqrt{g}), q_2 = \partial_2(\ln\sqrt{g}),$$

Where Ψ_2 has explicit form and is defined in [30]. Note that q_k do not depend on t.

We obtain the following equation from [30]:

$$\Delta(K) = \frac{1}{b(t)}(g\partial_1 a^1 + g\partial_2 a^2 + 2gq_k a^k - 2g\Psi_2 - \frac{g}{V}(M_{11}^4 + M_{22}^4) - \frac{g}{V^2}W_2^{(b)}), \qquad (2.1.4)$$

Where $M_{11}^4, M_{22}^4, W_2^{(b)}$ have explicit forms and are defined in [30].

Therefore we obtain:

$$\dot{\Delta}(K) = \frac{g}{b}(\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q_k^{(b)}\dot{a}^k - \dot{\Psi}_2^{(b)}), \qquad (2.1.5)$$

where $\dot{\Psi}_2^{(b)} = q_0^{(b)}\dot{c} - P_0(\dot{a}^1, \dot{a}^2, \partial_i \dot{a}^j)$. P_0 has explicit form. Notice that $q_k^{(b)} \in C^{m-3,\nu}$, $q_0^{(b)} \in C^{m-3,\nu}$ and do not depend on t.

Lemma 2.1.1. *Let the following conditions hold:*

1) metric tensor in R^3 satisfies the conditions: $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ such that $a^k(t), \partial_i a^k(t), \dot{a}^k(t), \partial_i \dot{a}^k(t)$ are continuous by $t, \forall t \in [0, t_0]$, $a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ such that $a^i(t) \in C^{m-2,\nu}, \partial_k a^i(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\exists t_* > 0$ such that for all $t \in [0, t_*)$ $P_0 \in C^{m-3,\nu}$ and the following inequality holds:

$$\|P_0(\dot{a}^1_{(1)}, \dot{a}^2_{(1)}) - P_0(\dot{a}^1_{(2)}, \dot{a}^2_{(2)})\|_{m-2,\nu} \le K_9(t)(\|\dot{a}^1_{(1)} - \dot{a}^1_{(2)}\|_{m-1,\nu} + \|\dot{a}^2_{(1)} - \dot{a}^2_{(2)}\|_{m-1,\nu}),$$

where for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \in [0, t_0)$ the following inequality holds: $K_9(t) < \varepsilon$.

The proof follows from [30].

§2.2. Deduction the formulas of $\Delta(H)$ and $\dot{\Delta}(H)$.

The formula of mean curvature is:

$$2H = g^{ij}b_{ij} = g^{11}b_{11} + g^{22}b_{22}, 2H(t) = g^{ij}(t)b_{ij}(t). \qquad (2.2.1)$$

Then we have:

$$2\Delta(H) = g^{ij}(t)b_{ij}(t) - g^{ij}b_{ij}. \qquad (2.2.2)$$

We use the following formulas:

$$g^{11}(t) = \frac{g_{22}(t)}{g(t)}, g^{22}(t) = \frac{g_{11}(t)}{g(t)}, g^{12}(t) = g^{21}(t) = -\frac{g_{12}(t)}{g(t)}. \qquad (2.2.3)$$

Then we have:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}(t)b_{11}(t) + g_{11}(t)b_{22}(t) - g_{12}(t)(b_{12}(t) + b_{21}(t)) - 2g(t)H). \qquad (2.2.4)$$

Consider the following formula:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}(t)b_{11}(t) + g_{11}(t)b_{22}(t) - g_{12}(t)(b_{12}(t) + b_{21}(t)) - 2g(t)H). \tag{2.2.5}$$

Use the formulas:

$$g_{ij}(t) = g_{ij} + \Delta(g_{ij}), b_{ij}(t) = b_{ij} + \Delta(b_{ij}), g(t) = g + \Delta(g). \tag{2.2.6}$$

Then we obtain the equation:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}b_{11}(t) + g_{11}b_{22}(t) - g_{12}(b_{12}(t) + b_{21}(t)) +$$
$$\Delta(g_{22})b_{11}(t) + \Delta(g_{11})b_{22}(t) - \Delta(g_{12})(b_{12}(t) + b_{21}(t)) - 2gH - 2\Delta(g)H). \tag{2.2.7}$$

Therefore we get the equation:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}b_{11} + g_{11}b_{22} - g_{12}(b_{12} + b_{21}) + g_{22}\Delta(b_{11}) + g_{11}\Delta(b_{22}) -$$
$$g_{12}(\Delta(b_{12}) + \Delta(b_{21})) + \Delta(g_{22})b_{11} + \Delta(g_{11})b_{22} - \Delta(g_{12})(b_{12} + b_{21}) +$$
$$\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2gH - 2\Delta(g)H). \tag{2.2.8}$$

Simplifying we obtain the equation:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}\Delta(b_{11}) + g_{11}\Delta(b_{22}) - g_{12}(\Delta(b_{12}) + \Delta(b_{21})) + \Delta(g_{22})b_{11} + \Delta(g_{11})b_{22} -$$
$$\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2\Delta(g)H). \tag{2.2.9}$$

Then we have:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}\Delta(b_{11}) + g_{11}\Delta(b_{22}) - g_{12}(\Delta(b_{12}) + \Delta(b_{21})) + V\Delta(g_{22}) + V\Delta(g_{11}) -$$
$$\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2\Delta(g)H). \tag{2.2.10}$$

We use the formula:

$$\Delta(b_{ij}) = \partial_i(a^k)b_{jk} + M^4_{ij}. \tag{2.2.11}$$

We can write the following:

$$\Delta(g_{ii}) = \partial_i(a^i)g_{ii} + M^5_{ii}. \tag{2.2.12}$$

Therefore we have

$$\Delta(H) = \frac{1}{2g(t)}(Vg_{22}\partial_1(a^1) + Vg_{11}\partial_2(a^2) + Vg_{22}\partial_2(a^2) + Vg_{11}\partial_1(a^1) +$$
$$g_{22}M^4_{11} + g_{11}M^4_{22} + VM^5_{22} + VM^5_{11} - g_{12}(\Delta(b_{12}) + \Delta(b_{21}))$$
$$-\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2\Delta(g)H). \tag{2.2.13}$$

Denote

$$\Psi_4 = g_{22}M^4_{11} + g_{11}M^4_{22} + VM^5_{22} + VM^5_{11} - g_{12}(\Delta(b_{12}) + \Delta(b_{21}))$$
$$-\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})). \quad (2.2.14)$$

Then we obtain the following equation

$$\Delta(H) = \frac{1}{2g(t)}(Vg_{22}\partial_1(a^1) + Vg_{11}\partial_2(a^2) + Vg_{22}\partial_2(a^2) + Vg_{11}\partial_1(a^1) + \Psi_4 - 2\Delta(g)H). \quad (2.2.15)$$

The equation takes the form

$$\Delta(H) = \frac{1}{2g(t)}(V(g_{11} + g_{22})(\partial_1(a^1) + \partial_2(a^2)) + \Psi_4 - 2\Delta(g)H). \quad (2.2.16)$$

We use formulas (2.1.20) and (2.1.21). Then we have

$$\Delta(H) = \frac{1}{2g(t)}(V(g_{11} + g_{22})(\partial_1 a^1 + \partial_2 a^2) + \Psi_4 -$$
$$4gH(\partial_1 a^1 + \partial_2 a^2 + q_k a^k - \Psi_2)). \quad (2.2.17)$$

Using the formula

$$2Hg = V(g_{11} + g_{22})$$

we get the following equation

$$\Delta(H) = \frac{1}{2g(t)}((-2gH(\partial_1 a^1 + \partial_2 a^2) - 4Hgq_k a^k + 4Hg\Psi_2 + \Psi_4). \quad (2.2.18)$$

Therefore

$$\Delta(H) = \frac{Hg}{g(t)}(-\partial_1 a^1 - \partial_2 a^2 - 2q_k a^k + 2\Psi_2 + \frac{\Psi_4}{2Hg}). \quad (2.2.19)$$

Differentiating by t we have

$$\dot\Delta(H) = \frac{Hg}{g(t)}(-\partial_1 \dot a^1 - \partial_2 \dot a^2 - 2q_k \dot a^k + 2\dot\Psi_2 + \frac{\dot\Psi_4}{2Hg}) -$$
$$\frac{Hg\dot g(t)}{(g(t))^2}(-\partial_1 a^1 - \partial_2 a^2 - 2q_k a^k + 2\Psi_2 + \frac{\Psi_4}{2Hg}). \quad (2.2.20)$$

Therefore we obtain:

$$\dot\Delta(H) = H(-\partial_1 \dot a^1 - \partial_2 \dot a^2 - q^{(h)}_k \dot a^k + \dot\Psi^{(h)}_2), \quad (2.2.21)$$

where $\dot\Psi^{(h)}_2 = q^{(h)}_0 \dot c - P^{(h)}_0(\dot a^1, \dot a^2, \partial_i \dot a^j)$. Notice that $q^{(h)}_k \in C^{m-3,\nu}$, $q^{(h)}_0 \in C^{m-3,\nu}$ and do not depend on t.

Lemma 2.2.1. *Let the following conditions hold:*

1) metric tensor in R^3 *satisfies the conditions:* $\exists M_0 = const > 0$ *such that* $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ *such that* $a^k(t), \partial_i a^k(t), \dot{a}^k(t), \partial_i \dot{a}^k(t)$ *are continuous by* $t, \forall t \in [0, t_0]$, $a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ *such that* $a^i(t) \in C^{m-2,\nu}, \partial_k a^i(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\exists t_* > 0$ *such that for all* $t \in [0, t_*)$ $P_0^{(h)} \in C^{m-3,\nu}$ *and the following inequality holds:*

$$\|P_0^{(h)}(\dot{a}^1_{(1)}, \dot{a}^2_{(1)}) - P_0^{(h)}(\dot{a}^1_{(2)}, \dot{a}^2_{(2)})\|_{m-2,\nu} \leq K_{10}(t)(\|\dot{a}^1_{(1)} - \dot{a}^1_{(2)}\|_{m-1,\nu} + \|\dot{a}^2_{(1)} - \dot{a}^2_{(2)}\|_{m-1,\nu}),$$

where for any $\varepsilon > 0$ *there exists* $t_0 > 0$ *such that for all* $t \in [0, t_0)$ *the following inequality holds:* $K_{10}(t) < \varepsilon$.

The proof follows from construction of function $P_0^{(h)}$ and lemmas of §7 and §8.

Notice the following formula:

$$\Delta(H) = \frac{1}{2g(t)}(g_{22}\Delta(b_{11}) + g_{11}\Delta(b_{22}) -$$
$$g_{12}(\Delta(b_{12}) + \Delta(b_{21})) + V\Delta(g_{22}) + V\Delta(g_{11}) -$$
$$\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2\Delta(g)H). \qquad (2.2.22)$$

Therefore we get the following formula

$$\Delta(H) = \frac{1}{2g(t)}(-2gH(\partial_1 a^1 + \partial_2 a^2) - 4Hgq_k a^k + 4Hg\Psi_2 + \Psi_4). \qquad (2.2.23)$$

Therefore we obtain:

$$\dot{\Delta}(H) = -\frac{\dot{g}(t)}{2(g(t))^2}(g_{22}\Delta(b_{11}) + g_{11}\Delta(b_{22}) -$$
$$g_{12}(\Delta(b_{12}) + \Delta(b_{21})) + V\Delta(g_{22}) + V\Delta(g_{11}) -$$
$$\Delta(g_{22})\Delta(b_{11}) + \Delta(g_{11})\Delta(b_{22}) - \Delta(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) - 2\Delta(g)H) +$$
$$\frac{1}{2g(t)}(g_{22}\dot{\Delta}(b_{11}) + g_{11}\dot{\Delta}(b_{22}) - g_{12}(\dot{\Delta}(b_{12}) + \dot{\Delta}(b_{21})) + V\dot{\Delta}(g_{22}) + V\dot{\Delta}(g_{11}) -$$
$$\dot{\Delta}(g_{22})\Delta(b_{11}) + \dot{\Delta}(g_{11})\Delta(b_{22}) - \dot{\Delta}(g_{12})(\Delta(b_{12}) + \Delta(b_{21})) +$$
$$\Delta(g_{22})\dot{\Delta}(b_{11}) + \Delta(g_{11})\dot{\Delta}(b_{22}) - \Delta(g_{12})(\dot{\Delta}(b_{12}) + \dot{\Delta}(b_{21})) - 2\dot{\Delta}(g)H). \qquad (2.2.24)$$

Hence we can write the following

$$\dot{\Delta}(H) = \frac{1}{2g(t)}(-2gH(\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2) - 4Hgq_k \dot{a}^k + 4Hg\dot{\Psi}_2 + \dot{\Psi}_4)$$
$$-\frac{\dot{g}(t)}{2(g(t))^2}(-2gH(\partial_1 a^1 + \partial_2 a^2) - 4Hgq_k a^k + 4Hg\Psi_2 + \Psi_4). \qquad (2.2.25)$$

We obtain

$$\Delta(H) = \frac{1}{2g(t)}(-2gH(\partial_1 a^1 + \partial_2 a^2 + q_k^{(h)} a^k) + \Psi_2^{(h)}). \tag{2.2.26}$$

Therefore we get

$$\dot{\Delta}(H) = \frac{1}{2g(t)}(-2gH(\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q_k^{(h)} \dot{a}^k) + \dot{\Psi}_2^{(h)})$$

$$-\frac{\dot{g}(t)}{2(g(t))^2}(-2gH(\partial_1 a^1 + \partial_2 a^2 + q_k^{(h)} a^k) + \Psi_2^{(h)}). \tag{2.2.27}$$

§2.3. Deduction the formulas of deformations preserving the sum of principal radii of curvature.

We have the formula

$$\Delta(\frac{H}{K}) = \frac{H(t)}{K(t)} - \frac{H}{K}. \tag{2.3.1}$$

Therefore we obtain the equation of $Ch-$deformation preserving the sum of principal radii of curvature.

$$\Delta(H) = \frac{H}{K}\Delta(K). \tag{2.3.2}$$

Using formulas from §2.1. and §2.2. we have

$$\partial_1 \dot{a}^1 + \partial_2 \dot{a}^2 + q_k^{(c)} \dot{a}^k = \dot{\Psi}_2^{(c)}, \tag{2.3.3}$$

where $\dot{\Psi}_2^{(c)} = q_0^{(c)}\dot{c} - P_0^{(c)}$. Note that $q_k^{(c)}$ do not depend on t, $P_0^{(c)}(\dot{a}^1, \dot{a}^2, \partial_i \dot{a}^j)$. Notice that $q_k^{(c)} \in C^{m-3,\nu}$, $q_0^{(c)} \in C^{m-3,\nu}$ and do not depend on t.

Lemma 2.3.1. *Let the following conditions hold:*

1) metric tensor in R^3 satisfies the conditions: $\exists M_0 = const > 0$ such that $\|\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$, $\|\partial^2\tilde{a}_{\alpha\beta}\|_{m,\nu} < M_0$.

2) $\exists t_0 > 0$ such that $a^k(t), \partial_i a^k(t), \dot{a}^k(t), \partial_i \dot{a}^k(t)$ are continuous by $t, \forall t \in [0, t_0]$, $a^k(0) \equiv 0, \partial_i a^k(0) \equiv 0$.

3) $\exists t_0 > 0$ such that $a^i(t) \in C^{m-2,\nu}, \partial_k a^i(t) \in C^{m-3,\nu}$, $\forall t \in [0, t_0]$.

Then $\exists t_* > 0$ such that for all $t \in [0, t_*)$ $P_0^{(c)} \in C^{m-3,\nu}$ and the following inequality holds:

$$\|P_0^{(c)}(\dot{a}^1_{(1)}, \dot{a}^2_{(1)}) - P_0^{(c)}(\dot{a}^1_{(2)}, \dot{a}^2_{(2)})\|_{m-2,\nu} \le K_{15}(t)(\|\dot{a}^1_{(1)} - \dot{a}^1_{(2)}\|_{m-1,\nu} + \|\dot{a}^2_{(1)} - \dot{a}^2_{(2)}\|_{m-1,\nu}),$$

where for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t \in [0, t_0)$ the following inequality holds: $K_{15}(t) < \varepsilon$.

The proof follows from construction of function $P_0^{(c)}$ and lemmas of §7 and §8 of [30].

The equation (2.3.3) determines deformations of surface preserving the sum of principal radii of curvature with condition of $G-$deformation.

§3. Proof of theorems 1 and 2.

We have the following equation systems of elliptic type

a) for $ChG-$deformations:

$$\partial_2\dot{a}^1 - \partial_1\dot{a}^2 + p_k\dot{a}^k = \dot{\Psi}_1,$$

$$\partial_1\dot{a}^1 + \partial_2\dot{a}^2 + q_k^{(c)}\dot{a}^k = \dot{\Psi}_2^{(c)}, \tag{3.1a}$$

where we use (2.1.2) and (2.3.3). $\dot{\Psi}_2^{(c)} = q_0^{(c)}\dot{c} - P_0^{(c)}$. Note that $q_k^{(c)}$ do not depend on t.

b) for $HG-$deformations:

$$\partial_2\dot{a}^1 - \partial_1\dot{a}^2 + p_k\dot{a}^k = \dot{\Psi}_1,$$

$$\partial_1\dot{a}^1 + \partial_2\dot{a}^2 + q_k^{(h)}\dot{a}^k = \dot{\Psi}_2^{(h)}, \tag{3.1b}$$

where we use (2.1.2) and (2.2.21). $\dot{\Psi}_2^{(h)} = q_0^{(h)}\dot{c} - P_0^{(h)}$. Note that $q_k^{(h)}$ do not depend on t.

c) for $AG-$deformations:

$$\partial_2\dot{a}^1 - \partial_1\dot{a}^2 + p_k\dot{a}^k = \dot{\Psi}_1,$$

$$\partial_1\dot{a}^1 + \partial_2\dot{a}^2 + q_k\dot{a}^k = \dot{\Psi}_2, \tag{3.1c}$$

where we use (2.1.2) and (2.1.3). $\dot{\Psi}_2$ is defined in [30]. Note that q_k do not depend on t.

For theorem 1, we reduce (3.1a), (3.1b) and (3.1c) with boundary-value condition (1.4) by the methods form [30] to the following form of desired boundary-value problem:

$$\partial_{\bar{z}}\dot{w} + A\dot{w} + B\bar{\dot{w}} + E(\dot{w}) = \dot{\Psi}, \qquad Re\{\overline{\lambda}\dot{w}\} = \dot{\varphi} \quad on \quad \partial D, \tag{3.14}$$

where $\dot{\Psi}$, $\dot{\varphi}$, E have explicit form and are defined in a similar way as it was made in [30], $\lambda = \lambda_1 + i\lambda_2$, $|\lambda| \equiv 1$, $\lambda, \dot{\varphi} \in C^{m-2,\nu}(\partial D)$.

We use estimations of norms for obtained functions from §2, §3, §7 of [30]. Formulas of functions $\dot{W}_1, \dot{W}_2, \dot{\Psi}_2$, the estimations for norms of these functions are presented in §8 of [30]. From article [30], we also use the following lemmas: 2.1, 2.2, 2.3, 3.1, 3.2, 5.1, 6.2.1, 7.1, 7.2, 7.3, 8.3.1, 8.3.2, and theorems: 1, 9.1, 9.2.

Therefore proof of theorem 1 follows from using similar reasonings for proving theorem 1 from article [30].

We obtain the proof of theorem 2 by using methods for proving theorem 1 from article [32].

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